THE STRUCTURE OF CONVOLUTION MEASURE ALGEBRAS([1])

BY
JOSEPH L. TAYLOR

Let G be a locally compact abelian topological group, $L_1(G)$ the algebra of Haar integrable functions on G under convolution multiplication, and $M(G)$ the algebra of regular Borel measures on G under convolution multiplication. $L_1(G)$ has been extensively studied and has been the motivation behind many of the results in Banach algebra theory; its maximal ideal space is $G^\wedge$, the character group of G. $M(G)$ is a far more complex algebra and a workable characterization of its maximal ideal space has been elusive. In this paper we attempt to give such a characterization. In analogy with $L_1(G)$, we represent the maximal ideal space of $M(G)$ as the set $S^\wedge$ of all semicharacters on a compact topological semigroup S. The methods used are applicable to a large class of Banach algebras which we have labeled convolution measure algebras. $L_1(G)$ and $M(G)$ are convolution measure algebras as is the measure algebra on any locally compact topological semigroup.

The Banach space structure of a convolution measure algebra is that of a complex L-space. This allows us to use the L-space theory developed by Kakutani, Cunningham, and others. In §1 we identify the adjoint space of a complex L-space as the space of all continuous functions on some compact Hausdorff space. This follows from similar results of Cunningham for real L-spaces. For this paper we have chosen to use the abstract definition of complex L-space. A detailed development of the results of §1 based on a concrete definition of L-space may be found in the author's dissertation [11].

In §2, we define the concept of convolution measure algebra and prove, using the results of §1, that the maximal ideal space of a commutative convolution measure algebra $\mathfrak{M}$ may be represented as the set $S^\wedge$ of all semicharacters on some compact topological semigroup S. In §3, we investigate the structures of $\mathfrak{M}$, S, and $S^\wedge$. We identify a subset H of $S^\wedge$ on which the Gelfand transform of each element of $\mathfrak{M}$ attains its maximum modulus. In §4, we discuss briefly three special

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([1]) The results of this paper comprise a portion of the author's doctoral dissertation which was written at Louisiana State University under the direction of Professor Pasquale Porcelli. Some of these results we represented by title to the meeting of the Society held January 23–27, 1964, under the title *The maximal ideal space of the convolution measure algebra on a compact group*. This research was supported in part by United States Air Force Contract AFOSR 68–83 and the United States Army Research Office (Durham).

convolution measure algebras associated with a locally compact group G. These are: $L_1(G)$, $M(G)$, and the convolution algebras discussed by Arens and Singer in [1].

1. **L-spaces and L-homomorphisms.** In this section we summarize a portion of the theory of L-spaces as developed in [2] and [3], and prove some theorems concerning homomorphisms of such spaces.

A real L-space is a lattice ordered real Banach space $\mathfrak{L}$, such that $\mu, \nu \in \mathfrak{L}$, $\mu \geqq 0$, $\nu \geqq 0$ implies $\|\mu + \nu\| = \|\mu\| + \|\nu\|$ and $\mu, \nu \in \mathfrak{L}$ with $\mu \wedge \nu = 0$ implies $\|\mu + \nu\| = \|\mu - \nu\|$. If $\mathfrak{L}$ is a Banach space of real bounded measures (not necessarily countably additive) on a Boolean algebra, such that $\mathfrak{L}$ is clodse under sup and inf, then it is easily seen that $\mathfrak{L}$ is a real L-space. Conversely, Kakutani has shown that every real L-space may be represented as such a space of measures (cf. [3]).

Rieffel has mentioned in [5] an extension of the concept of L-space to complex Banach spaces. We restate this as follows:

DEFINITION 1.1. A complex L-space is a partially ordered complex Banach space $\mathfrak{M}$, such that the real subspace $\mathfrak{M}_r$ generated by $\{\mu \in \mathfrak{M}: \mu \geqq 0\}$ is a real L-space and

(a) if $\mu \in \mathfrak{M}$ there exist unique elements $\mathrm{Re}(\mu)$, $\mathrm{Im}(\mu) \in \mathfrak{M}_r$, such that $\mu = \mathrm{Re}(\mu) + i\ \mathrm{Im}(\mu)$, and

(b) if $|\mu| = \bigvee \{\mathrm{Re}(e^{i\theta}\mu): 0 \leqq \theta < 2\pi\}$, then $\|\mu\| = \|\,|\mu|\,\|$ for each $\mu \in \mathfrak{M}$.

An L-subspace $\mathfrak{N}$ of a complex L-space $\mathfrak{M}$ is a closed subspace, such that $0 < \nu \leqq \mu$, $\mu \in \mathfrak{N}$, and $\nu \in \mathfrak{M}$ implies $\nu \in \mathfrak{N}$, and $\mu \in \mathfrak{N}$ implies $|\mu| \in \mathfrak{N}$. It can be shown that an L-subspace of a complex L-space is again a complex L-space. If $\mathfrak{N}$ is an L-subspace of $\mathfrak{M}$ then $\mathfrak{N}^{\perp}$ is the space of all $\mu \in \mathfrak{M}$, such that $|\mu| \wedge |\nu| = 0$ for each $\nu \in \mathfrak{N}$. $\mathfrak{N}^{\perp}$ is again an L-subspace of $\mathfrak{M}$ and $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}^{\perp}$.

If $\mathfrak{M}$ is a complex Banach space of measures on a Boolean algebra, such that the real measures in $\mathfrak{M}$ form a lattice and $\mathrm{Re}(\mu)$, $\mathrm{Im}(\mu) \in \mathfrak{M}$ whenever $\mu \in \mathfrak{M}$, then clearly $\mathfrak{M}$ is a complex L-space. Here $|\mu|$ is the ordinary total variation of the measure μ. In analogy with Kakutani's results, it can be shown that every complex L-space may be represented as such a space of measures.

Throughout the remainder of this section, $\mathfrak{M}$ will denote a fixed complex L-space.

DEFINITION 1.2. R will denote the algebra of bounded operators T on $\mathfrak{M}$, such that $T\mathfrak{N} \subset \mathfrak{N}$ for every L-subspace $\mathfrak{N}$ of $\mathfrak{M}$.

The following theorem is an adaption to the complex case of several results in the theory of real L-spaces.

THEOREM 1.1. *There is a compact, extremaly disconnected Hausdorff space X, an order-preserving isometry $\mu \to \mu_X$ of $\mathfrak{M}$ into $M(X)$ (the space of regular Borel measures on X), an isometry $F \to f$ of $\mathfrak{M}^*$ onto $C(X)$, and an isomorphism-isometry $T \to f$ of R onto $C(X)$, such that*

$$F(\mu) = \int f d\mu_X \quad \text{and} \quad (T\mu)_X(V) = \int_V f d\mu_X$$

for each Borel set V of X.

The proof follows from §8 and §9 of [2] which yields similar results for real L-spaces. The extension to the complex case is straightforward. A detailed proof may be found in [11]. To each L-subspace $\mathfrak{N}$ of $\mathfrak{M}$ there corresponds a unique projection operator $P \in R$, called the L-projection on $\mathfrak{N}$, such that $P\mathfrak{M} = \mathfrak{N}$ and $(I-P)\mathfrak{M} = \mathfrak{N}^{\perp}$. The space X may be described as the Stone space of the Boolean algebra of L-projections P, as in [2], or it may be described as the maximal ideal space of R, as in [11].

The imbedding $\mu \to \mu_X$ of Theorem 1.1 preserves order and norm, and, hence, preserves all L-space properties. Thus we may identify $\mathfrak{M}$ with its image $\mathfrak{M}_X$ in $M(X)$, $\mathfrak{M}^*$ with $C(X)$, $\mathfrak{M}^{**}$ with $M(X)$, and R with the algebra of operators determined by $C(X)$ via integration. X will be called the standard domain of $\mathfrak{M}$. It can be shown that $\mathfrak{M}_X$ is the L-subspace of $M(X)$ consisting of all $\lambda \in M(X)$ such that $\lambda(V) = 0$ for each Borel set V of X with empty interior (cf. [11, Theorem 1.12]).

Note that for $\mu \in \mathfrak{M}$, $|\mu|_X = |\mu_X|$, where

$$|\mu_X|(U) = \sup \left\{ \sum_{i=1}^{n} |\mu_X(V_i \cap U)| : \{V_i\}_{i=1}^{n} \text{ is a Borel subdivision of } X \right\}$$

is the total variation of the measure μ_X. If $\mu, \nu \in \mathfrak{M}$ then ν is in the L-subspace generated by μ if and only if ν_X is absolutely continuous with respect to μ_X. Similarly, $|\mu| \wedge |\nu| = 0$ if and only if ν_X is purely singular with respect to μ_X.

We complete this section with some homomorphism theorems which are essential to obtaining the results of §2. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be complex L-spaces with standard domains X_1 and X_2 respectively. By Theorem 1.1, we may identify $\mathfrak{M}_1$ with $(\mathfrak{M}_1)_{X_1}$ and $\mathfrak{M}_2$ with $(\mathfrak{M}_2)_{X_2}$ and drop the use of the subscripts.

Definition 1.3. If θ is a bounded linear map from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, then θ will be called an L-homomorphism if the following conditions are satisfied:

(1) if $0 \leqq \mu \in \mathfrak{M}_1$, then $\|\theta\mu\| = \|\mu\|$;

(2) if $0 \leqq \mu \in \mathfrak{M}_1$, then $\theta\mu \geqq 0$; and

(3) if $\mu \in \mathfrak{M}_1$, $\nu \in \mathfrak{M}_2$, $\mu \geqq 0$, and $0 \leqq \nu \leqq \theta\mu$, then there exists $\omega \in \mathfrak{M}_1$, $0 \leqq \omega \leqq \mu$, such that $\theta\omega = \nu$.

Lemma 1.1. *If θ is an L-homomorphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, $\mathfrak{N}$ is an L-subspace of $\mathfrak{M}_2$, and $\mathfrak{N}' = \{\mu \in \mathfrak{M}_1 : \theta|\mu| \in \mathfrak{N}\}$, then $\mathfrak{N}'$ is an L-subspace of $\mathfrak{M}_1$ and $(\mathfrak{N}')^{\perp} = (\mathfrak{N}^{\perp})'$.*

Proof. That $\mathfrak{N}'$ is an L-subspace follows from (2) of Definition 1.3. If $\mu \in (\mathfrak{N}')^{\perp}$, then $|\mu| \in (\mathfrak{N}')^{\perp}$ and $\theta|\mu| \geqq 0$. Either $\theta|\mu| \in \mathfrak{N}^{\perp}$ or there exists nonzero $\nu \in \mathfrak{N}$ i.h) $\leqq \nu \leqq \theta|\mu|$. In the latter case, (3) of Definition 1.3 yields an element $\omega \in \mathfrak{M}_1$, such that $0 \leqq \omega \leqq |\mu|$ and $\theta\omega = \nu$. Then $\omega \in (\mathfrak{N}')^{\perp}$; however $\theta\omega = \nu \in \mathfrak{N}$ and so $\omega \in \mathfrak{N}'$. It follows that $\omega = 0$ and $\nu = 0$. The resulting contradiction shows that $\theta|\mu| \in \mathfrak{N}^{\perp}$. Hence $(\mathfrak{N}')^{\perp} \subset (\mathfrak{N}^{\perp})'$. The reverse containment is clear.

LEMMA 1.2. *If Y and Z are compact Hausdorff spaces and α is a continuous map from Y to Z, then the equation*

$$\theta\mu(V) = \mu(\alpha^{-1}(V)) \text{ for } V \text{ a Borel set of } Z$$

defines an L-homomorphism θ from $M(Y)$ to $M(Z)$.

Proof. θ clearly satisfies (1) and (2) of Definition 1.3. If $\mu \in M(Y)$, $\mu \geqq 0$, $\nu \in M(Z)$, and $0 \leqq \nu \leqq \theta\mu$, then by the Radon-Nikodym Theorem, there exists a Borel function f on Z, such that $0 \leqq f \leqq 1$ and $\nu(V) = \int_V f\, d\theta\mu$ for each Borel set V of Z. If we set $\theta^* f(y) = f(\alpha(y))$, then $\theta^* f$ is a Borel function on Y, $0 \leqq \theta^* f \leqq 1$, and $\theta\omega = \nu$ where $\omega(V) = \int_V \theta^* f\, d\mu$. Hence (3) of Definition 1.3 holds and θ is an L-homomorphism.

THEOREM 1.2. *If θ is a bounded linear map from $\mathfrak{M}_1$ to $\mathfrak{M}_2$, then the following statements are equivalent:*

(a) *θ is an L-homomorphism;*

(b) *there exists a continuous map α from X_1 to X_2, such that $\theta\mu(V) = \mu(\alpha^{-1}(V))$ for each Borel set V of X_2;*

(c) *θ^*, the adjoint map of θ, is a homomorphism of the algebra $C(X_2)$ into the algebra $C(X_1)$, $\theta^* \bar{f} = \overline{\theta^* f}$, and $\theta^* 1 = 1$.*

Proof. (b) and (c) are clearly equivalent. That (b) implies (a) follows from Lemma 1.2. Hence it suffices to prove that (a) implies (b).

If $\mathfrak{N}$ is an L-subspace of $\mathfrak{M}_2$, let P and P' be the L-projections on $\mathfrak{N}$ and $\mathfrak{N}'$ respectively (cf. Lemma 1.1). By Lemma 1.1, if $\mu \in \mathfrak{M}_1$ then $\theta P'\mu \in \mathfrak{N}$ and $\theta(I - P')\mu \in \mathfrak{N}^\perp$. It follows that

$$P\theta\mu = P[\theta P'\mu + \theta(I - P')\mu] = \theta P'\mu.$$

If $\mu \in \mathfrak{M}_1$, $\mu \geqq 0$, then by (1) of Definition 1.3,

$$P'\mu(X_1) = \| P'\mu \| = \| \theta P'\mu \| = \| P\theta\mu \| = P\theta\mu(X_2).$$

It follows that $P'\mu(X_1) = P\theta\mu(X_2)$ for any $\mu \in \mathfrak{M}_1$.

On the basis of Theorem 1.1 we may conclude that there is a one-to-one correspondence $P \to U$ between the Boolean algebra of all L-projections P on $\mathfrak{M}_2$ and all open-compact subsets U of X_2, such that $P\mu(X_2) = \mu(U)$, and a similar correspondence $P' \to U'$ between the L-projections on $\mathfrak{M}_1$ and the open-compact subsets of X_1. Hence, by the above paragraph, the correspondence $\mathfrak{N} \to \mathfrak{N}'$ of Lemma 1.1 determines a map $U \to U'$ from all open-compact subsets of X_2 to open-compact subsets of X_1. This map clearly satisfies $(U_1 \cap U_2)' = U_1' \cap U_2'$, $(X_2 \setminus U)' = X_1 \setminus U'$, and $\theta\mu(U) = \mu(U')$ for each $\mu \in \mathfrak{M}_1$. The map $U \to U'$ generates a continuous map α from X_1 to X_2, such that $\alpha^{-1}(U) = U'$ for each open-compact subset U of X_2. This map is described as follows: If $x \in X_1$, let $\alpha(x) = \bigcap \{U : x \in U'\}$; since X_2 is extremally disconnected it has a basis of open-compact sets U; it follows from this and the properties of the map $U \to U'$ that $\alpha(x)$ is a single point of X_2 for each $x \in X_1$, $\alpha^{-1}(U) = U'$ for U open-compact in X_2, and α is continuous. Now since $\theta\mu(U) = \mu(U') = \mu(\alpha^{-1}(U))$, it follows that α is the required map. This completes the proof.

COROLLARY. *If an L-homomorphism θ is one-to-one, then θ is generated by a homeomorphism of X_1 into X_2 and is an isometry.*

Condition (3) in the definition of an L-homomorphism may be replaced by a formally weaker condition, namely:

(3′) There exists a set $\mathfrak{F}$ of positive elements of $\mathfrak{M}_1$, such that every positive $\mu \in \mathfrak{M}_1$ is the norm limit of linear combinations, with positive coefficients, of elements of $\mathfrak{F}$, and if $\mu \in \mathfrak{F}$, $\nu \in \mathfrak{M}_2$, and $0 \leqq \nu \leqq \theta\mu$, then there exists a sequence $\{\mu_i\}_{i=1}^{\infty} \subset \mathfrak{M}_1$, $0 \leqq \mu_i \leqq \mu$ for each i, and $\lim_i \theta\mu_i = \nu$.

THEOREM 1.3. *If θ is a bounded linear map from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ which satisfies* (1) *and* (2) *of Definition* 1.3 *and* (3′) *as stated above, then θ is an L-homomorphism.*

Proof. We shall show that θ satisfies (3) of Definition 1.3. Let $\mathfrak{F}$ be the subset of $\mathfrak{M}_1$ given by condition (3′). If $\mu \in \mathfrak{M}_1$ and $\mu \geqq 0$, then by (3′) there exist measures $\{\mu_{ij}\}_{j,i=1}^{n_i,\infty} \subset \mathfrak{F}$ and numbers $a_{ij} \geqq 0$, such that if $\mu_i = \sum_{j=1}^{n_i} a_{ij}\mu_{ij}$ then $\lim_i \mu_i = \mu$. If $\nu \in \mathfrak{M}_2$ and $0 \leqq \nu \leqq \theta\mu$ we set $\nu_i = \nu \wedge \theta\mu_i$; then $0 \leqq \nu_i \leqq \theta\mu_i$ and $\lim_i \nu_i = \nu \wedge \theta\mu = \nu$. Similarly, we can write

$$\nu_i = \sum_{j=1}^{n_i} a_{ij}\nu_{ij}$$

for some collection of measures $\nu_{ij} \in \mathfrak{M}_2$ with $0 \leqq \nu_{ij} \leqq \theta\mu_{ij}$. These statements follow from the fact that the real measures in $\mathfrak{M}_2$ form a lattice ordered Banach space.

By (3′) there exists for each i and j a sequence $\{\omega_{ijk}\}_{k=1}^{\infty} \subset \mathfrak{M}_1$ such that $0 \leqq \omega_{ijk} \leqq \mu_{ij}$ and $\lim_k \theta\omega_{ijk} = \nu_{ij}$. Let $\omega_{ik} = \sum_{j=1}^{n_i} a_{ij}\omega_{ijk}$. Then $0 \leqq \omega_{ik} \leqq \mu_i$ and $\lim_k \theta\omega_{ik} = \nu_i$. Since $\lim_i \nu_i = \nu$, there exists a diagonal sequence $\{\omega_{ik_i}\}_{i=1}^{\infty}$, such that $\lim_i \theta\omega_{ik_i} = \nu$.

We now apply some results of Porcelli concerning weak convergence in spaces of measures (cf. Theorems 3.2 and 4.3 of [4]). Since $\{\mu_i\}_{i=1}^{\infty}$ converges in norm and $0 \leqq \omega_{ik_i} \leqq \mu_i$ for each i, it follows that $\{\omega_{ik_i}\}_{i=1}^{\infty}$ has a weakly convergent subsequence. Let ω be the limit of such a subsequence. Then $0 \leqq \omega \leqq \mu$ and $\theta\omega = \nu$. Thus θ satisfies (3) of Definition 1.3. This completes the proof.

2. **Convolution measure algebras.** Throughout this section $\mathfrak{M}$ will denote a complex L-space with standard domain X. We shall consider $\mathfrak{M}$ as a subspace of $M(X)$ in accordance with Theorem 1.1.

DEFINITION 2.1. A convolution measure algebra is a complex L-space $\mathfrak{M}$

together with a multiplication $\cdot$ on $\mathfrak{M}$, such that $(\mathfrak{M}, \cdot)$ forms a Banach algebra and the following conditions are satisfied:

(1) If $\mu, \nu \in \mathfrak{M}$, $\mu \geqq 0$, $\nu \geqq 0$, then $\|\mu \cdot \nu\| = \|\mu\| \, \|\nu\|$.

(2) If $\mu, \nu \in \mathfrak{M}$, $\mu \geqq 0$, $\nu \geqq 0$, then $\mu \cdot \nu \geqq 0$.

(3) If μ, ν, and ω are positive elements of $\mathfrak{M}$ and $\omega \leqq \mu \cdot \nu$, then for each $\varepsilon > 0$ there exist sets $\{\mu_i\}_{i=1}^n$ and $\{\nu_j\}_{j=1}^m$ of positive elements of $\mathfrak{M}$ and a set of numbers $\{a_{ij}\}_{i,j=1}^{n,m} \subset [0,1]$, such that $\sum_{i=1}^n \mu_i \leqq \mu$; $\sum_{j=1}^m \nu_j \leqq \nu$, and $\|\omega - \sum_{ij} a_{ij}\mu_i \cdot \nu_j\| < \varepsilon$.

Let S be any locally compact topological semigroup (in particular S may be a group). Convolution in $M(S)$ is a multiplication $\cdot$ defined by the equation

$$\int f(s)\, d(\mu \cdot \nu)(s) = \iint f(st)\, d\mu(s)\, d\nu(t)$$

for each bounded Borel function f on S. It is well known that $M(S)$ is a Banach algebra under this multiplication (cf.[10]).

THEOREM 2.1. *If S is a locally compact topological semigroup, then $M(S)$ is a convolution measure algebra.*

Proof. (1) and (2) of Definition 2.1 clearly hold for convolution in $M(S)$.

To show that (3) holds, let μ, ν, and ω be positive measures in $M(S)$, such that $\omega \leqq \mu \cdot \nu$. By the Radon-Nikodym Theorem, there exists a Borel function f on S, such that $0 \leqq f \leqq 1$ and $\omega(V) = \int_V f\, d\mu \cdot \nu$ for each Borel set V of S. Let h be the Borel function on $S \times S$ defined by $h(s,t) = f(st)$. Clearly $0 \leqq h \leqq 1$. It follows that for each $\varepsilon > 0$ there exists a function k on $S \times S$ of the form

$$k = \sum_{i,j=1}^{n,m} a_{ij}\pi_{ij},$$

where $0 \leqq a_{ij} \leqq 1$, $\{U_i\}_{i=1}^n$ and $\{V_j\}_{j=1}^m$ are Borel subdivisions of S, π_{ij} is the characteristic function of $U_i \times V_j$, and $\int |h - k| d(\mu \times \nu) < \varepsilon$. Let $\mu_i(V) = \mu(V \cap U_i)$ and $\nu_j(V) = \nu(V \cap V_j)$ for each i and j and each Borel set V of S. Then

$$\sum_{i=1}^n \mu_i = \mu \quad \text{and} \quad \sum_{j=1}^m \nu_j = \nu.$$

Also, it follows from Fubini's Theorem and the definition of convolution that if g is any continuous function on S and $\rho = \sum_{i,j} a_{ij}(\mu_i \cdot \nu_j)$, then

$$\left| \int g\, d(\rho - \omega) \right| = \left| \iint g(st)(k(s,t) - h(s,t))\, d\mu(s) d\nu(t) \right| \leqq \|g\| \varepsilon.$$

Hence $\|\rho - \omega\| < \varepsilon$. This completes the proof.

DEFINITION 2.2. If $\mathfrak{M}$ is a complex L-space, let $\mathfrak{M} \otimes \mathfrak{M}$ denote the L-subspace of $M(X \times X)$ generated by the set of measures $\mu \times \nu$ for $\mu, \nu \in \mathfrak{M}$.

Let $\sum_{i=1}^\infty \rho_i$ be a convergent series of measures ρ_i in $M(X \times X)$ with each ρ

absolutely continuous with respect to some $\mu_i \times \nu_i$ for $\mu_i, \nu_i \in \mathfrak{M}$. Clearly $\sum_{i=1}^{\infty}\rho_i$ is absolutely continuous with respect to $\mu \times \nu$, where

$$\mu = \sum_{n=1}^{\infty} 2^{-n} \| \mu_i \|^{-1} |\mu_i| \text{ and } \nu = \sum_{n=1}^{\infty} 2^{-n} \| \nu_i \|^{-1} |\nu_i| .$$

It follows that $\mathfrak{M} \otimes \mathfrak{M}$ is precisely the set of all measures in $M(X \times X)$ each of which is absolutely continuous with respect to some $\mu \times \nu$ for $\mu, \nu \in \mathfrak{M}$. Also, by the Radon-Nikodym Theorem each positive measure in $\mathfrak{M} \otimes \mathfrak{M}$ is the norm limit of linear combinations, with positive coefficients, if measure of the form $\mu \times \nu$ for μ, ν positive measures in $\mathfrak{M}$.

LEMMA 2.1. *If $(\mathfrak{M}, \cdot)$ is a convolution measure algebra, then there exists an L-homomorphism θ from $\mathfrak{M} \otimes \mathfrak{M}$ to $\mathfrak{M}$ such that $\theta(\mu \times \nu) = \mu \cdot \nu$ for each $\mu, \nu \in \mathfrak{M}$.*

Proof. Fix $\mu, \nu \in \mathfrak{M}$; we define θ first on $\mathfrak{L}(\mu, \times \nu)$, the space of measures in $M(X \times X)$ which are absolutely continuous with respect to $\mu \times \nu$. Let $\{U_i\}_{i=1}^n$ and $\{V_j\}_{j=1}^m$ be Borel subdivisions of X and set $\mu_i(V) = \mu(V \cap U_i), \nu_j(V) = (V \cap V_j)$ Note that the collection $\{\mu_i \times \nu_j\}_{i,j=1}^{n;m}$ consists of mutually singular measures. If ρ is any measure of the form $\rho = \sum_{i,j} a_{ij} \mu_j \times \nu_j$, we set $\theta\rho = \sum_{i,j} a_{ij}\mu_i \cdot \nu_j$. Then

$$\begin{aligned} \| \theta\rho \| &\leqq \sum_{i,j} |a_{ij}| \, \|\mu_i \cdot \nu_j\| \leqq \sum_{i,j} |a_{ij}| \, \|\mu_j\| \, \|\nu_j\| \\ &= \sum_{i,j} |a_{ij}| \, \|\mu_i \times \nu_j\| = \left\| \sum_{i,j} a_{ij}\mu_i + \nu_j \right\| = \|\rho\| . \end{aligned}$$

A simple refinement argument shows that the definition of $\theta\rho$ is independent of the subdivisions $\{U_i\}_{i=1}^n$ and $\{V_j\}_{j=1}^m$ which are used in the representation of ρ. The collection of measures which can be represented in the above form for some $\{U_i\}_{i=1}^n$, $\{V_j\}_{j=1}^m$ and $\{a_{ij}\}_{i,j=1}^{n;m}$ comprises a dense subspace of $\mathfrak{L}(\mu \times \nu)$. θ as defined above is clearly linear and of norm one on this subspace. Hence θ may be extended to all of $\mathfrak{L}(\mu \times \nu)$. Clearly $\theta(\mu_1 \times \nu_1) = \mu_1 \cdot \nu_1$ if $\mu_1 \times \nu_1 \in \mathfrak{L}(\mu \times \nu)$ and θ is uniquely defined by this condition. If μ_1, ν_1, μ_2, $\nu_2 \in \mathfrak{M}$ and we carry out the above process for $\mathfrak{L}(\mu_1 \times \nu_1)$ and $\mathfrak{L}(\mu_2 \times \nu_2)$, obtaining maps θ_1 and θ_2 respectively, then θ_1 and θ_2 must agree on $\mathfrak{L}(\mu_1 \times \nu_1) \cap \mathfrak{L}(\mu_2 \times \nu_2)$. It follows that θ may be defined on all of $\mathfrak{M} \otimes \mathfrak{M}$ satisfying $\theta(\mu \times \nu) = \mu \cdot \nu$ for each μ, $\nu \in \mathfrak{M}$.

Note that (1) and (2) of Definition 2.1 imply that θ satisfies (1) and (2) of Definition 1.3. If we set $\mathfrak{F} = \{\mu \times \nu : \mu, \nu \in \mathfrak{M}, \mu \geqq 0, \nu \geqq 0\}$, then (3) of Definition 2.1 implies that $\mathfrak{F}$ and θ satisfy (3′) as used in Theorem 1.3. Hence, by that theorem, θ is an L-homomorphism.

LEMMA 2.2. *If Y is the standard domain of $\mathfrak{M} \otimes \mathfrak{M}$, then there is an isomorphism-isometry ϕ of $C(X \times X)$ into $C(Y)$, such that $\phi\bar{f} = \overline{\phi f}$, $\phi 1 = 1$, and $\int f d\rho = \int \phi f d\rho_Y$ for each $f \in C(X \times C(X \times X)$ and $\rho \in \mathfrak{M} \otimes \mathfrak{M}$.*

Proof. Let R be the algebra of bounded operators on $\mathfrak{M} \otimes \mathfrak{M}$ which leave all L-subspaces invariant. By Theorem 1.1, R is isomorphic-isometric to $C(Y)$ under a map $T \to f$, such that $(T\rho)_Y(V) = \int_V f \, d\rho_Y$. Each $h \in C(X \times X)$ determines an operator $T \in R$, such that $T\rho(U) = \int_U h \, d\rho$. The map $h \to T$ is an isomorphism-isometry of $C(X \times X)$ into R. ϕ is the composition of $h \to T$ and $T \to f$. Clearly ϕ has the required properties. The map ϕ is not generally onto; in fact, if it were then $X \times X$ and Y would be homeomorphic and $X \times X$ would be extremally disconnected. However, it can easily be shown that $X \times X$ is extremally disconnected if and only if X is finite.

DEFINITION 2.3. If $(\mathfrak{M}, \cdot)$ is a commutative convolution measure algebra, we denote by Δ the collection of all functions $f \in C(X)$, such that

$$\int f \, d(\mu \cdot \nu) = \int f \, d\mu \int f \, d\nu$$

for each μ, $\nu \in \mathfrak{M}$ and f is not identically zero.

Since $\mathfrak{M}^* = C(X)$, the maximal ideal space of $(\mathfrak{M}, \cdot)$ is, by definition, Δ with the weak-*topology of $\mathfrak{M}^*$.

LEMMA 2.3. *If $f \in C(X)$ then $f \in \Delta$ if and only if $\theta^* f \in \phi C(X \times X)$ and $\phi^{-1}\theta^* f(x, y) = f(x)f(y)$ for each $x, y \in X$, where θ and ϕ are as defined in Lemmas 2.1 and 2.2.*

Proof. If $f \in C(X)$ let h be the function in $C(X \times X)$ defined by $h(x, y) = f(x)f(y)$. By Lemma 2.2, $\phi h = \theta^* f$ if and only if $\int h \, d\rho = \int \phi h \, d\rho_Y = \int f \, d\theta\rho$ for each $\rho \in \mathfrak{M} \otimes \mathfrak{M}$. Since $\mathfrak{M} \otimes \mathfrak{M}$ is generated by the collection $\{\mu \times \nu : \mu, \nu \in \mathfrak{M}\}$, $\phi h = \theta^* f$ if and only if $\int f d(\mu \cdot \nu) = \int h \, d(\mu \times \nu) = \int f d\mu \int f d\nu$ for each μ, $\nu \in \mathfrak{M}$, i.e., if and only if $f \in \Delta$.

COROLLARY. (a) $1 \in \Delta$,

(b) *if $f \in \Delta$, then $\bar{f} \in \Delta$, and*

(c) *if f and $h \in \Delta$ then $fh \in \Delta$ provided $fh \neq 0$.*

Proof. By Theorem 1.2 and Lemmas 2.1 and 2.2, $\phi^{-1}\theta^*$ is a homomorphism, $\phi^{-1}\theta^* \bar{f} = \overline{\phi^{-1}\theta^* f}$ and $\phi^{-1}\theta^* 1 = 1$. Hence the corollary follows from the characterization of Δ in Lemma 2.3.

DEFINITION 2.4. If S is a topological semigroup, then a semicharacter on S is a continuous function f of norm less than or equal to one on S, which is not identically zero and such that $f(st) = f(s)f(t)$ for each $s, t \in S$. The collection of semicharacters on S will be denoted by $S^{\wedge}$.

THEOREM 2.2. *If $(\mathfrak{M}, \cdot)$ is a commutative convolution measure algebra, then there is a compact, abelian topological semigroup S and a continuous map σ of X onto S, such that*

(a) *$S^{\wedge}$ separates points in S, and*

(b) *the map $h \to h\sigma$ is a one to one map of $S^{\wedge}$ onto Δ.*

Proof. Let Λ be the closed linear span of Δ in $C(X)$. By the corollary to Lemma 2.3, Λ is a subalgebra of $C(X)$ which is closed under conjugation and contains the constant functions. If $x, y \in X$ set $x \sim y$ if $f(x) = f(y)$ for every $f \in \Lambda$. If S is the factor space of X modulo the equivalence relation $\sim$ and σ is the natural map from X onto S, then S is a compact Hausdorff space, σ is continuous, and σ induces an isomorphism-isometry $h \to h\sigma$ of $C(S)$ onto Λ.

If $f \in \Delta$ then by Lemma 2.3, $\theta^* f \in \phi C(X \times X)$ and $\phi^{-1}\theta^* f$ is constant on subsets of $X \times X$ of the form $\sigma^{-1}(s) \times \sigma^{-1}(t)$ for $s, t \in S$. This allows us to define a map γ from $C(S)$ to $C(S \times S)$ in the following manner. If $h \in C(S)$ then $h\sigma \in \Lambda$ and $\phi^{-1}\theta^*(h\sigma)$ is constant on each $\sigma^{-1}(s) \times \sigma^{-1}(t)$ for s, $t \in S$. Hence there exists a function $\gamma h \in C(S \times S)$ such that

$$\gamma h(\sigma(x), \sigma(y)) = \phi^{-1}\theta^*(h\sigma)(x, y).$$

It follows from the properties of θ^*, ϕ, and σ that γ is a bounded homomorphism of $C(S)$ into $C(S \times S)$, $\gamma \bar{f} = \overline{\gamma f}$, and $\gamma 1 = 1$. Note that if $h \in C(S)$, then $h\sigma \in \Delta$ if and only if $\gamma h(s,t) = h(s)h(t)$.

Each point (s,t) of $S \times S$ determines a homomorphism δ of $C(S)$ into the complex numbers through the formula $\delta h = \gamma h(s,t)$. Each such homomorphism arises from a point $st \in S$, such that $\delta h = h(st)$; i.e., there is a map $(s,t) \to st$ from $S \times S$ to S, such that $h(st) = \gamma h(s,t)$ for each $h \in C(S)$. Note that $h\sigma \in \Delta$ if and only if $h(st) = h(s)h(t)$, and since the closed linear span of Δ is Λ, this completely determines the map $(s,t) \to st$. It follows that $(s,t) \to st$ is a commutative, associative, and continuous multiplication on S. Under this multiplication S is a compact abelian topological semigroup and $S^\wedge = \{h \in C(S) : h\sigma \in \Delta\}$. Since the closed linear span of Δ is Λ, the closed linear span of $S^\wedge$ is $C(S)$ and $S^\wedge$ separates points in S. This completes the proof.

We shall call S the structure semigroup of $(\mathfrak{M}, \cdot)$.

Definition 2.5. A C-homomorphism of a convolution measure algebra $(\mathfrak{M}_1, \cdot)$ into a convolution measure algebra $(\mathfrak{M}_2, \cdot)$ is a bounded homomorphism of $(\mathfrak{M}_1, \cdot)$ into $(\mathfrak{M}_2, \cdot)$ which is also an L-homomorphism of $\mathfrak{M}_1$ into $\mathfrak{M}_2$.

Theorem 2.3. *If $(\mathfrak{M}, \cdot)$ is a commutative convolution measure algebra, X the standard domain of $\mathfrak{M}$, S the structure semigroup of $(\mathfrak{M}, \cdot)$ and σ the natural map from X to S, then the map $\mu \to \mu_S$ of $(\mathfrak{M}, \cdot)$ into $M(S)$ defined by $\mu_S(V) = \mu(\sigma^{-1}(V))$ is a C-homomorphism; also:*

(a) *$\int h \, d\mu_S = \int h\sigma \, d\mu$ for $\mu \in \mathfrak{M}$ and $h \in C(S)$,*

(b) *the image $\mathfrak{M}_S$ of $\mathfrak{M}$ is weak-* dense in $M(S)$, and*

(c) *$\mu \to \mu_S$ is an isometry if and only if $(\mathfrak{M}, \cdot)$ is semisimple.*

Proof. By Lemma 1.2, $\mu \to \mu_S$ is an L-homomorphism of $\mathfrak{M}$ into $M(S)$. Clearly $\int h \, d\mu_S = \int h\sigma \, d\mu$ for each $h \in C(S)$. To see that $\mu \to \mu_S$ is an algebraic homomorphism, note that if $h \in S^\wedge$ and $\mu, \nu \in \mathfrak{M}$, then

$$\int h\, d(\mu \cdot \nu)_S = \int h\sigma\, d(\mu \cdot \nu) = \int h\sigma\, d\mu \int h\sigma\, d\nu$$

$$= \int h\, d\mu_S \int h\, d\nu_S = \iint h(st)\, d\mu_S(s)\, d\nu_S(t)$$

$$= \int h\, d\mu_S \cdot \nu_S.$$

Hence

$$(\mu \cdot \nu)_S = \mu_S \cdot \nu_S.$$

Since $\mathfrak{M}$ separates points in $C(X)$, $\mathfrak{M}_S$ separates points in $C(S)$; hence $\mathfrak{M}_S$ is weak-* dense in $M(S)$.

The map $\mu \to \mu_S$ is clearly one to one if and only if Δ separates points in $\mathfrak{M}$, i.e., if and only if $(\mathfrak{M}, \cdot)$ is semisimple. By the corollary to Theorem 1.2, $\mu \to \mu_S$ is one to one if and only if it is an isometry. This completes the proof.

On the basis of Theorems 2.2 and 2.3 we may identify the maximal ideal space of a commutative convolution measure algebra $(\mathfrak{M}, \cdot)$ with $S^\wedge$, the set of semicharacters on the structure semigroup S of $(\mathfrak{M}, \cdot)$. If $\mu \in \mathfrak{M}$, then the Gelfand transform $\mu^\wedge$ of μ is the function defined on $S^\wedge$ by $\mu^\wedge(h) = \int h\, d\mu_S$. The Gelfand topology on $S^\wedge$ is the weak topology generated by the algebra of functions $\mu^\wedge$ for $\mu \in \mathfrak{M}$. $S^\wedge$ is locally compact in this topology and is compact if and only if $(\mathfrak{M}, \cdot)$ has an identity. If $\mu \in \mathfrak{M}$ then $\mu^\wedge \in C_0(S^\wedge)$, the algebra of continuous functions on $S^\wedge$ which vanish at infinity.

3. **The structure of S and $S^\wedge$.** Let $(\mathfrak{M}, \cdot)$ be a commutative semisimple convolution measure algebra and S the structure semigroup of $(\mathfrak{M}, \cdot)$. By Theorem 2.3, we may identify $\mathfrak{M}$ with $\mathfrak{M}_S$ and drop the use of the subscript. Hence $\mathfrak{M}$ will be considered an L-subalgebra of $M(S)$. As in §1, R will denote the algebra of bounded operators on $\mathfrak{M}$ which leave L-subspaces invariant.

In Definition 2.4 we insist that $S^\wedge$ consists of nonzero homomorphisms of S into the unit disc. The set $S^\wedge \cup \{0\}$ of all continuous homomorphisms of S into the unit disc is clearly a semigroup with identity under pointwise multiplication. $S^\wedge$ itself may not be closed under this multiplication. For example, let $S = \{a, b, c\}$ where $a^2 = a$, $b^2 = b$, and $ab = ac = bc = c^2 = c$; here $S^\wedge = \{f, g, h\}$ where $f(a) = f(b) = f(c) = 1$, $g(a) = 1$, $g(b) = g(c) = 0$, $h(b) = 1$, and $h(a) = h(c) = 0$; in this case $gh = 0 \notin S^\wedge$. However, in the important case where S has an identity, $S^\wedge$ is clearly closed under pointwise multiplication and is itself a semigroup with identity.

THEOREM 3.1. *S has an identity if and only if $\mathfrak{M}$ has an approximate identity of norm one, i.e., a net $\{\mu_\alpha\}_{\alpha \in \mathfrak{A}} \subset \mathfrak{M}$, such that $\|\mu_\alpha\| = 1$ for each α and $\lim_\alpha \mu_\alpha \cdot \mu = \mu$ in norm for each $\mu \in \mathfrak{M}$.*

Proof. If S has an identity e, let $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$ be a neighborhood basis at e. Since $\mathfrak{M}$ is a weak-* dense L-subspace of $M(S)$, there is a positive measure μ_α, of norm one, concentrated on each U_α. It is easily seen that the net $\{\mu_\alpha\}_{\alpha \in \mathfrak{A}}$ is an approximate identity for $\mathfrak{M}$.

Conversely, if $\mathfrak{M}$ has an approximate identity $\{\mu_\alpha\}_{\alpha \in \mathfrak{A}}$ of norm one, then $\{\mu_\alpha\}_{\alpha \in \mathfrak{A}}$ clusters to some element $\nu \in M(S)$ in the weak-* topology of $M(S)$. It follows that $\|\nu\| = 1$ and $\nu \cdot \mu = \mu$ for each $\mu \in \mathfrak{M}$. Thus $\int f d\mu = \int f d\nu \cdot \mu = \int f d\nu \int f d\mu$ for each $\mu \in \mathfrak{M}$ and $f \in S^\wedge$. Therefore $\int f d\nu = 1$ for each $f \in S^\wedge$. However, $\|\nu\| = 1$ and $\|f\| \leqq 1$ for $f \in S^\wedge$ and hence each $f \in S^\wedge$ is identically one on the carrier of ν. Since $S^\wedge$ separates points in S, it follows that ν is concentrated on a single point e such that $f(e) = 1$ for $f \in S^\wedge$. This implies that e is an identity for S.

The following lemma is a straightforward consequence of Theorems 1.1, 2.2 and 2.3.

Lemma 3.1. *If $T \in R$ then T is an algebraic homomorphism of $\mathfrak{M}$ if and only if there exists $f \in S^\wedge \cup \{0\}$ such that $T\mu(V) = \int_V f d\mu$.*

Since, by Lemma 3.1, $S^\wedge \cup \{0\}$ may be identified with a semigroup of operators in R, it inherits two natural topologies from R: the strong and weak operator topologies. These are the topologies of pointwise convergence of operators on $\mathfrak{M}$ in the norm and weak topologies of $\mathfrak{M}$, respectively. On $S^\wedge$ the weak topology coincides with the Gelfand topology. Pointwise multiplication is not generally continuous in this topology (cf. §4). However, for fixed $g \in S^\wedge$ it is easily seen that the map $f \to gf$ is weakly continuous. Multiplication is continuous in the strong topology on $S^\wedge \cup \{0\}$. This suggests that the strong topology may be a more natural topology to use; however, the author has not investigated this point. In the case where $\mathfrak{M} = L_1(G)$ for some locally compact abelian group G, the two topologies coincide (cf. §4). Unless otherwise specified we shall use the weak topology on $S^\wedge \cup \{0\}$ from here on.

Definition 3.1. An ideal of $\mathfrak{M}$ which is also an L-subspace of $\mathfrak{M}$ will be called an L-ideal. If $\mathfrak{I}$ is an L-ideal of $\mathfrak{M}$ and $\mathfrak{I}^\perp$ is a subalgebra, then $\mathfrak{I}$ will be called a prime L-ideal. An ideal J of S, such that $S \setminus J$ is a subsemigroup, will be called a prime ideal.

Theorem 3.2. *If $\mathfrak{I}$ is an L-subspace of $\mathfrak{M}$, then the following statements are equivalent:*

(a) *$\mathfrak{I}$ is a prime L-ideal;*

(b) *the projection $P \in R$, which projects on $\mathfrak{I}^\perp$, is a homomorphism;*

(c) *there is an idempotent $\pi \in S^\wedge$, such that $\mathfrak{I} = \{\mu \in \mathfrak{M}: \int \pi d|\mu| = 0\}$;*

(d) *there is an open-compact prime ideal $J \subset S$, such that $\mathfrak{I} = \{\mu \in \mathfrak{M}: \text{carrier}(\mu) \subset J\}$.*

Proof. This follows readily from Theorem 2.2, Lemma 3.1, and the fact that carrier $(\mu \cdot \nu) \subset$ carrier (μ) carrier (ν) for $\mu, \nu \in M(S)$. Here π is the function in $S^\wedge$ corresponding to P as in Lemma 3.1 and is the characteristic function of $S \setminus J$.

The above theorem establishes a one to one correspondence between prime L-ideals of $\mathfrak{M}$, idempotent homomorphisms in R, idempotents in $S^\wedge$, and open-compact prime ideals of S. The maximal groups in $S^\wedge$ may be characterized as follows: if $\pi \in S^\wedge$ is an idempotent, then the maximal group $G^\wedge(\pi)$ containing π is the set of all $h \in S^\wedge$, such that $|h| = \pi$. If π is the characteristic function of $S \setminus J$, where J is an open-compact prime ideal of S, then $G^\wedge(\pi)$ is isomorphic to the character group of the kernel (minimal ideal) G of $S \setminus J$ through the restriction map $h \to h|_G$. If H is the union of all maximal groups $G^\wedge(\pi)$ in $S^\wedge$, then $H = \{h \in S^\wedge : |h(s)| = 0 \text{ or } 1 \text{ for each } s \in S\}$ and H is made up of building blocks which may be identified with groups of characters. Much of the pathology of convolution measure algebras arises when H is a proper subset of $S^\wedge$.

LEMMA 3.2. *If g is a bounded Borel function on S and $g(st) = g(s)g(t)$ for $s, t \in S$, then there exists $f \in S^\wedge \cup \{0\}$ such that $f = g$ except on a set N of μ-measure zero for each $\mu \in \mathfrak{M}$.*

Proof. We define $T\mu(V) = \int_V g \, d\mu$ for $\mu \in \mathfrak{M}$ and V a Borel set of S. Then $T \in R$ and T is a homomorphism. Hence, by Lemma 3.1, there exists $f \in S^\wedge$, such that $T\mu(V) = \int_V f \, d\mu$. The conclusion follows.

COROLLARY. *If S_0 is an open (closed) subsemigroup of S whose complement is an ideal, then the closure (interior) of S_0 is either empty or an open-compact subsemigroup whose complement is an ideal.*

Proof. If g is the characteristic function of S_0, then g clearly satisfies the hypothesis of Lemma 3.2 and, hence, there is a $\pi \in S^\wedge \cup \{0\}$ such that $\pi = g$ almost everywhere with respect to each measure in $\mathfrak{M}$. It follows readily that π is idempotent and is the characteristic function of the closure of S_0 if S_0 is open and the interior of S_0 if S_0 is closed. The conclusion follows immediately.

Note that if $f \in S^\wedge$, then $\{s \in S : f(s) \neq 0\}$ is an open subsemigroup whose complement is an ideal and, hence, its closure is an open-compact subsemigroup whose complement is an ideal. We shall call the closure of $\{s \in S : f(s) \neq 0\}$ the support of f.

LEMMA 3.3. *If $f \in S^\wedge$, then $|f| \in S^\wedge$ and there exists a unique $h \in H$ such that $f = |f|h$ and h has the same support as f.*

Proof. We set $h_1(s) = |f(s)|^{-1} f(s)$ if $f(s) \neq 0$, $h_1(s) = 0$ if $f(s) = 0$. Then h_1 is a bounded Borel function and $h_1(st) = h_1(s)h_1(t)$ for each $s, t \in S$. By Lemma 3.2 there exists $h \in S^\wedge$, such that $h = h_1$ except on a set of μ-measure zero for each $\mu \in \mathfrak{M}$. Since h is continuous, it follows that $h \in H$ and $f = |f|h$. Clearly h has the same support as f and is unique with this property.

If $r \in S^\wedge$, $r \geqq 0$, and z is a complex number with $\mathrm{Re}(z) > 0$, then the function

r^z is again in $S^\wedge$. The map $z \to r^z$ is a vector-valued analytic function from $\{z\colon \operatorname{Re}(z) > 0\}$ into $S^\wedge$. If $H = S^\wedge$, then $r \in S^\wedge$, $r \geqq 0$, implies that r is an idempotent and the map $z \to r^z$ is constant. However, if $H \neq S^\wedge$, then there exist functions $r \in S^\wedge$, $r \geqq 0$, for which this map is nontrivial.

THEOREM 3.3. *If $\mu \in \mathfrak{M}$ then μ attains its maximum modulus on H.*

Proof. Since $\mu \in C_0(S)$, it attains its maximum modulus at some point, say f, in $S^\wedge$. Let $f = |f|h$ as in Lemma 3.3. We define $\xi(z) = \int |f|^z h \, d\mu = \mu^\wedge(|f|^z h)$ for $\operatorname{Re}(z) > 0$. ξ is analytic on $\{z\colon \operatorname{Re}(z) > 0\}$; however, ξ attains its maximum at $z = 1$. Hence ξ is a constant; i.e., $\mu^\wedge(|f|^z h) = \mu^\wedge(f)$ for all z such that $\operatorname{Re}(z) > 0$.

Let $k(s) = \lim_n |f|^n(s)$ and note that $k(s) = 0$ if $|f|(s) < 1$, $k(s) = 1$ if $|f|(s) = 1$. It follows that k is a bounded Borel function and $k(st) = k(s)k(t)$ for each $s, t \in S$. Hence there exists $\pi \in S^\wedge$ (π must be idempotent), such that $\pi = k$ except on a set of μ-measure zero for each $\mu \in \mathfrak{M}$. Then $\pi h \in H$ and $\mu^\wedge(\pi h) = \lim_n \mu^\wedge(|f|^n h) = \mu^\wedge(f)$ by the Lebesgue dominated convergence theorem. Hence $\mu^\wedge$ attains its maximum modulus at $\pi h \in H$. This completes the proof.

The Šilov boundary of $\mathfrak{M}$ is the smallest subset of $S^\wedge$ which is closed in the Gelfand topology and on which each $\mu^\wedge \in \mathfrak{M}^\wedge$ attains its maximum modulus. A point $f \in S^\wedge$ is called a strong boundary point if for each open subset U of $S^\wedge$ which contains f, there exists a function ζ in the norm closure of $\mathfrak{M}^\wedge$ in $C_0(S^\wedge)$ such that ζ attains its maximum modulus at f but not at any point of $S^\wedge \setminus U$. Theorem 3.2 shows that the closure of H contains the Šilov boundary. A similar argument shows that H itself contains the set of strong boundary points. Unfortunately, when multiplication is not continuous in $S^\wedge \cup \{0\}$, H may not be closed in $S^\wedge$. This is the case when $\mathfrak{M} = M(G)$ (cf. §4). Note, however, that H is closed in $S^\wedge$ in the strong topology since multiplication is continuous in the strong topology.

Let K be the union of all maximal groups in S. K is a compact subsemigroup of S. If $s \in K$, then s^{-1} denotes the inverse of s in the maximal group containing s. The map $s \to s^{-1}$ is a continuous isomorphism of K onto K (cf. [12, p. 98]).

LEMMA 3.4. *If $s \in S$, then $s \in K$ if and only if $|f(s)| = 0$ or 1 for each $f \in S^\wedge$.*

Proof. If s is in some group, then f restricted to that group is either a character or zero for $f \in S^\wedge$. Thus $|f(s)| = 0$ or 1.

To prove the converse, note that if $\mathfrak{O}(s)$ denotes the closure of $\{s^n\colon n = 1, 2, \cdots\}$ then $\mathfrak{O}(s)$ is a compact semigroup and therefore has a kernel Q which is a group. If p is the identity of Q, then clearly $|f(s)| = 0$ implies $f(p) = 0$ and $|f(s)| = 1$ implies $f(p) = 1$ for $f \in S^\wedge$. Thus if $|f(s)| = 0$ or 1 for each $f \in S^\wedge$, then $f(sp) = f(s)f(p) = f(s)$ for each $f \in S^\wedge$, and $sp = s$ since $S^\wedge$ separates points in S. However, $sp \in Q \subset K$. This completes the proof.

DEFINITION 3.2. If $\mu \in M(K)$ and V is a Borel set of K, then we define $\tilde{\mu}(V) = \bar{\mu}(V^{-1})$.

The map $\mu \to \tilde{\mu}$ is clearly an involution on the algebra $M(K)$. The next theorem shows how the symmetry of $\mathfrak{M}$ is related to the semigroup K and the involution $\sim$.

THEOREM 3.4. *If $\mu \in \mathfrak{M}$, then $\overline{\mu^\wedge} \in \mathfrak{M}^\wedge$ if and only if $\mu \in M(K)$ and $\tilde{\mu} \in \mathfrak{M}$. In this case $\tilde{\mu}^\wedge = \overline{\mu^\wedge}$. $\mathfrak{M}$ is symmetric if and only if $K=S$ and $\mathfrak{M}$ is closed under the involution $\sim$.*

Proof. Suppose $\mu \in \mathfrak{M}$ and there exists $\nu \in \mathfrak{M}$, such that $\overline{\mu^\wedge} = \nu^\wedge$. We shall show that carrier $(\mu) \subset K$. If not, then there exists $s \in$ carrier $(\mu) \setminus K$ and $f \in S^\wedge$, such that $0 < |f(s)| < 1$. Let $U = \{t \in S: 0 < |f(t)| < 1\}$. There exists $h \in S^\wedge$, such that $\int_U |f| h \, d\mu \neq 0$, since U meets carrier (μ). We define, for $\mathrm{Re}(z) > 0$, $\xi(z) = \int |f|^z h \, d\mu = \int_U |f|^z h \, d\mu + C$, where $C = \lim_n \int |f|^n h \, d\mu$. Since $\xi(1) \neq C$ and $\lim_n \xi(n) = C$, ξ is an analytic function which is nonconstant. Hence $\bar{\xi}$ is not analytic. However, $\bar{\xi}(z) = \overline{\mu^\wedge}(|f|^z h) = \nu^\wedge(|f|^z h)$ and ξ must be analytic. The resulting contradiction shows that carrier $(\mu) \subset K$. The rest of the theorem now follows readily.

The author has been unable to determine whether in general, or even when $\mathfrak{M} = M(G)$, it is true that $\mathfrak{M} \cap M(K)$ is closed under the involution $\sim$. If the answer to this question were affirmative, then Theorem 3.4 could be simplified to read: "If $\mu \in \mathfrak{M}$ then $\overline{\mu^\wedge} \in \mathfrak{M}^\wedge$ if and only if $\mu \in M(K)$. $\mathfrak{M}$ is symmetric if and only if $K = S$."

4. **Examples.** Let G be a locally compact abelian group and $G^\wedge$ the character group of G. In this section we discuss three types of convolution measure algebras associated with G.

i. $L_1(G)$. The algebra $L_1(G)$ of all Haar integrable functions on G under convolution multiplication is clearly a convolution measure algebra. It is well known that the maximal ideal space of $L_1(G)$ may be identified with $G^\wedge$. Thus, it is easily seen that, for $L_1(G)$, S is the Bohr compactification of G and $S^\wedge = G^\wedge$ (the Bohr compactification of G is the unique compact group whose dual group is isomorphic to $G^\wedge$). The weak or Gelfand topology induced on $G^\wedge$ by $L_1(G)$ coincides with the compact-open topology of $G^\wedge$ as a space of functions on G. It follows from this that the weak and strong topologies discussed in §3 coincide on $G^\wedge$.

ii. The algebras of Arens and Singer. Let G_+ be a closed subsemigroup of G such that the interior of G_+ is dense in G_+. Let A be the subalgebra of $L_1(G)$ consisting of those functions in $L_1(G)$ which vanish outside G_+. A is clearly an L-subalgebra of $L_1(G)$. Arens and Singer in [1] show that the maximal ideal space of A is $G_+^\wedge$, the space of all semicharacters on G_+. It follows that there is a continuous isomorphism of G_+ onto a dense subsemigroup of S, the structure semigroup of A. This map is determined by the corresponding isomorphism of $S^\wedge$ onto $G_+^\wedge$. Thus S is a compactification in some sense of the semigroup G_+.

Theorem 3.1 of [1] shows that the set H discussed in §3 corresponds to the set of all restrictions to G_+ of characters of G. Theorem 4.6 of [1] shows that this is the Šilov boundary of the maximal ideal space of A. As in the case of $L_1(G)$, the weak topology of $G_+^\wedge$ coincides with the compact-open topology and, hence, with the strong topology.

In the particular case when G is the group of integers and G_+ the semigroup of nonnegative integers, the semigroup $G_+^\wedge = S^\wedge$ becomes the closed unit disc under multiplication, H becomes the unit circle, A is the algebra of absolutely convergent sequences under convolution, and the Gelfand transform takes a sequence to the corresponding analytic function on the disc.

Note that in the preceding examples S turns out to be a compactification of a more appropriate semigroup. We suspect that this may be the case in general and hope to investigate the question in a later paper.

iii. $M(G)$. Let $M(G)$ be the algebra of all bounded regular Borel measures on G under convolution multiplication and let S be the structure semigroup of $M(G)$. The structure of S is largely undeterminded in this case for even the simplest nondiscrete groups. However, there are a few observations which can be made. Each $\chi \in G^\wedge$ determines a complex homomorphism F of $M(G)$ through the formula

$$F(\mu) = \int \chi \, d\mu.$$

Hence there is a natural imbedding $\chi \to h_\chi$ of $G^\wedge$ into $S^\wedge$, such that

$$\int \chi \, d\mu = \int h_\chi d\mu_S$$

for each $\mu \in M(G)$. It can easily be shown that $\chi \to h_\chi$ is an isomorphism of $G^\wedge$ onto $G^\wedge(1) = \{h \in S^\wedge : |h| = 1\}$. Since $G^\wedge(1)$ is isomorphic to the character group of the kernel of S, it follows that the kernel of S is the Bohr compactification of G. Similarly, it can be shown that the maximal group at the identity in S is the Bohr compactification of (G, d), where (G, d) is G with the discrete topology (that S has an identity in this case follows from Theorem 3.1 and the fact that $M(G)$ has an identity of norm one).

It is well known that if G is nondiscrete, then $M(G)$ is not symmetric (cf. [6, p. 107]). The next theorem, in conjunction with Theorem 3.4, shows that the asymmetry of $M(G)$ is related to the structure of S.

THEOREM 4.1. *If G is nondiscrete, then there is a compact subset V of S which carries nonzero continuous measures in $(M(G))_S$ such that every continuous function of norm less than or equal to one on V is the restriction to V of a semi-character in $S^\wedge$. It follows that H is a proper subset of $S^\wedge$ and K is a proper subset of S.*

Proof. Hewitt and Kakutani have shown that every nondiscrete locally compact abelian group G contains a Cantor set P, such that if $M_c(P)$ denotes the space of continuous measures in $M(P)$, then every linear functional of norm less than or equal to one on $M_c(P)$ is the restriction to $M_c(P)$ of a complex homomorphism of $M(G)$ (cf. [6, Theorem 5.4]). It follows that if V is the smallest closed subset of S containing carrier (μ_S) for each $\mu \in M_c(P)$, then V satisfies the conditions of the theorem.

The method used in Theorem 5.4.1 of [6] shows that there are points of $S^\wedge \setminus H$ which are in the Šilov boundary. Given a linear functional L on $M_c(P)$ with $\|L\| \leqq 1$, a $\mu \in M_c(P)$, and $\varepsilon > 0$, the set $H(\mu, \varepsilon)$ is defined to be the set of all complex homomorphisms h of $M(G)$ for which $|h(\mu) - L(\mu)| \leqq \varepsilon$. If $\{H(\mu_i, \varepsilon_i)\}_{i=1}^n$ is any finite collection of these sets, then Lemma 5.4.2 of [6] is used to show the existence of an $h \in \bigcap_i H(\mu_i, \varepsilon_i)$. This h is chosen to be any complex homomorphism at which the Gelfand transform of a certain measure attains its maximum. Thus h may be chosen from the Šilov boundary. It follows from the compactness of the sets $H(\mu, \varepsilon)$ and the Šilov boundary that there is an h_0 in the Šilov boundary which is in all $H(\mu, \varepsilon)$. Hence, $h_0 = L$ on $M_c(P)$. Clearly, if $\|L\| < 1$ then h_0 must correspond to a semicharacter in $S^\wedge \setminus H$.

The above discussion, together with Theorem 3.3, shows that if G is nondiscrete, then H is not closed in $S^\wedge$ and multiplication is not continuous in $S^\wedge$.

On p. 235 of [7] Rudin proposes three problems concerning $M(G)$:

(a) What is the Šilov boundary of the maximal ideal space of $M(G)$?

(b) Is there a subset of the maximal ideal space of $M(G)$ larger than the closure of $G^\wedge$, on which the restrictions of the Gelfand transforms are closed under conjugation?

(c) Characterize $R(G)$, where $R(G)$ is the set of all $\mu \in M(G)$ for which there exists $\nu \in M(G)$ such that $\overline{\mu^\wedge} = \nu^\wedge$.

Theorem 3.3 gives a partial result on problem (a). In addition the author has proved that when $G = D_2$, the countable product of two point groups, the closure of H is a proper subset of $S^\wedge$ and, hence, the Šilov boundary is a proper subset of $S^\wedge$. This result will appear in a later paper. We conjecture that the above is true for any nondiscrete group G.

In connection with problem (c), Theorem 3.4 shows that $R(G)$ is contained in $R_1(G)$, the algebra of measures $\mu \in M(G)$ for which μ_S is carried on K. We conjecture that $R(G) = R_1(G)$.

Problem (b) was settled in the affirmative by Simon for the case when G is the real line (cf. [8]).

Šreĭder in [9] has given a characterization of the maximal ideal space of $M(G)$ in terms of what he calls generalized characters. A generalized character is a collection of functions $\{f_\mu\}_{\mu \in M(G)}$ on G such that $f_\mu \in L_\infty(\mu)$ for each μ, $f_\mu(xy) = f_\mu(x)f_\mu(y)$ almost everywhere on $G \times G$ with respect to $\mu \times \mu$, and if ν is absolutely continuous with respect to μ, then $f_\nu = f_\mu$ almost everywhere with respect to ν. It can be shown that the set of generalized characters is closed under the obvious pointwise multiplication and that the resulting semigroup is isomorphic to our $S^{\wedge}$. However, Šreĭder's work does not indicate the existence of something like the underlying semigroup S.

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Louisiana State University,
Baton Rouge, Louisiana
Harvard University,
Cambridge, Massachusetts